Level 3

Part 1 Algebra (Polynomials and complex roots; Solving (systems of) equations; Optimization)

1. Given $f(z) = z^2 - 19z$, there are complex numbers z with the property that z, $f(z)$, and $f(f(z))$ are the vertices of a right triangle in the complex plane with a right angle at $f(z)$. There are positive integers m and n such that one such value of z is $m + \sqrt{n} + 11i$. Find $m + n$.

2. Let $a > 1$ and $x > 1$ satisfy $\log_a(\log_a(\log_a 2) + \log_a 24 - 128) = 128$ and $\log_a(\log_a x) = 256$. Find the remainder when x is divided by 1000.

3. It is given that the roots of the polynomial $P(z) = z^{2019} - 1$ can be written in the form $z_k = x_k + iy_k$ for $1 \leq k \leq 2019$. Let Q denote the monic polynomial with roots equal to $2x_k + iy_k$ for $1 \leq k \leq 2019$. Compute $Q(-2)$.

Part 2 Combinatorics (Expected value; Events with States; Recursion)

4. Let $SP_1P_2P_3EP_4P_5$ be a heptagon. A frog starts jumping at vertex S. From any vertex of the heptagon except E, the frog may jump to either of the two adjacent vertices. When it reaches vertex E, the frog stops and stays there. Find the number of distinct sequences of jumps of no more than 12 jumps that end at E.

5. Misha rolls a standard, fair six-sided die until she rolls 1-2-3 in that order on three consecutive rolls. The probability that she will roll the die an odd number of times is $\dfrac{m}{n}$ where m and n are relatively prime positive integers. Find $m + n$.

6. There are 100 lightbulbs $B_1, \ldots, B_{100}$ spaced evenly around a circle in this order. Additionally, there are 100 switches $S_1, \ldots, S_{100}$ such that for all $1 \leq i \leq 100$, switch S_i toggles the states of lights B_{i-1} and B_{i+1} (where here $B_{101} = B_1$). Suppose David chooses whether to flick each switch with probability $\frac{1}{2}$. What is the expected number of lightbulbs which are on at the end of this process given that not all lightbulbs are off?

Part 3 Geometry (Rotation/reflection/homothety; Laws of Sines and Cosines; Cyclic Quads and Ptolemy)

7. Let $\overline{AB}$ be a chord of a circle ω, and let P be a point on the chord $\overline{AB}$. Circle ω_1 passes through A and P and is internally tangent to ω. Circle ω_2 passes through B and P and is internally tangent to ω.

Circles ω_1 and ω_2 intersect at points P and Q. Line PQ intersects ω at X and Y. Assume that $AP = 5$, $PB = 3$, $XY = 11$, and $PQ^2 = \frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

8. Let $\triangle ABC$ have side lengths $AB = 30$, $BC = 32$, and $AC = 34$. Point X lies in the interior of $\overline{BC}$, and points I_1 and I_2 are the incenters of $\triangle ABX$ and $\triangle ACX$, respectively. Find the minimum possible area of $\triangle AI_1I_2$ as X varies along $\overline{BC}$.

9. The area of the smallest equilateral triangle with one vertex on each of the sides of the right triangle with side lengths $2\sqrt{3}$, 5, and $\sqrt{37}$, as shown, is $\frac{m\sqrt{p}}{n}$, where m, n, and p are positive integers, m and n are relatively prime, and p is not divisible by the square of any prime. Find $m + n + p$.

Part 4 Number Theory (Fermat Little, Euler's and Wilson theorems; Order of a number modulo p)

10. Find the least positive integer n such that when 3^n is written in base 143, its two right-most digits in base 143 are 01.

11. Find the least positive integer m such that $m^2 - m + 11$ is a product of at least four not necessarily distinct primes.