## Level 2.5

1. Find all pairs of integers $n$ and $k$, with $2 < k < n$, such that the binomial coefficients
$$\binom{n}{k-1}, \quad \binom{n}{k}, \quad \binom{n}{k+1}$$
form an increasing arithmetic sequence.

2. A sequence of positive integers with $a_1 = 1$ and $a_9 + a_{10} = 646$ is formed so that the first three terms are in geometric progression, the second, third, and fourth terms are in arithmetic progression, and, in general, for all $j \geq 1$, the terms $a_{2j-1}$, $a_{2j}$, $a_{2j+1}$ are in geometric progression, and the terms $a_{2j}$, $a_{2j+1}$, and $a_{2j+2}$ are in arithmetic progression. Find the greatest term in this sequence that is less than 1000.

3. For $0 < x < 1$, let
$$f(x) = (1+x)(1+x^4)(1+x^{16})(1+x^{64})(1+x^{256})\cdots.$$
Compute: $f^{-1}\left(\dfrac{8}{5f\left(\frac{3}{8}\right)}\right)$.

4. Compute $\dfrac{\lfloor \sqrt[4]{1} \rfloor \cdot \lfloor \sqrt[4]{3} \rfloor \cdot \lfloor \sqrt[4]{5} \rfloor \ldots \lfloor \sqrt[4]{2015} \rfloor}{\lfloor \sqrt[4]{2} \rfloor \cdot \lfloor \sqrt[4]{4} \rfloor \cdot \lfloor \sqrt[4]{6} \rfloor \ldots \lfloor \sqrt[4]{2016} \rfloor}$

5. Suppose that $a$ is positive, $\{a^{-1}\} = \{a^2\}$, and $2 < a^2 < 3$. Find the value of $a^{12} - 144a^{-1}$.

6. The function $f$ defined by
$$f(x) = \frac{ax+b}{cx+d},$$
where $a$, $b$, $c$, and $d$ are nonzero real numbers, has the properties $f(19) = 19$, $f(97) = 97$, and $f(f(x)) = x$ for all values of $x$ except $-d/c$. Find the unique number that is not in the range of $f$.

7. Find all complex numbers $z$ such that $|z-1| = |z+3| = |z-i|$.

8. A function $f$ is defined on the complex numbers by $f(z) = (a+bi)z$, where $a$ and $b$ are positive numbers. This function has the property that the image of each point in the complex plane is equidistant from that point and the origin. Given that $|a+bi| = 8$, find the value of $b^2$.

9. Find all values of $x$ such that
$$4x^2 - 6x - 41 + \frac{1}{2x^2 - 3x - 19} = 0.$$

10. Let
$$P(x) = x^4 + ax^3 + bx^2 + cx + d,$$
where $a$, $b$, $c$, and $d$ are constants. If $P(1) = 10$, $P(2) = 20$, and $P(3) = 30$, compute
$$\frac{P(12) + P(-8)}{10}.$$

11. Let $r_1$, $r_2$, $r_3$ be the 3 zeroes of the cubic polynomial $x^3 - x - 1 = 0$. Then, the expression
$$r_1(r_2 - r_3)^2 + r_2(r_3 - r_1)^2 + r_3(r_1 - r_2)^2$$

12. Compute the number of permutations $x_1 \ldots x_6$ of the integers $1, \ldots, 6$ such that $x_{i+1} \leq 2x_i$ for all $i$, $1 \leq i \leq 6$.

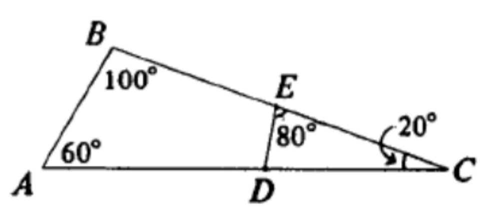
13. An $n$-sided die has the integers between 1 and $n$ (inclusive) on its faces. All values on the faces of the die are equally likely to be rolled. An 8-sided die, a 12-sided die, and a 20-sided die are rolled. Compute the probability that one of the values rolled is equal to the sum of the other two values rolled.

14. Taotao wants to buy a bracelet. The bracelets have 7 different beads on them, arranged in a circle. Two bracelets are the same if one can be rotated or flipped to get the other. If she can choose the colors and placement of the beads, and the beads come in orange, white, and black, how many possible bracelets can she buy?

15. A large elementary school class goes on a field trip to see a play. The front row of the theater has 11 seats. No boy wants to sit between 2 girls (or sit at the end of the row next to a girl) and no girl wants to sit between 2 boys (or sit at the end of the row next to a boy). In how many ways can the row of seats be assigned to boys and girls?

16. A paper equilateral triangle $ABC$ has side length 12. The paper triangle is folded so that vertex $A$ touches a point on side $\overline{BC}$ a distance 9 from point $B$. The length of the line segment along which the triangle is folded can be written as $\frac{m\sqrt{p}}{n}$, where $m$, $n$, and $p$ are positive integers, $m$ and $n$ are relatively prime, and $p$ is not divisible by the square of any prime. Find $m + n + p$.

17. In $\triangle ABC$, $E$ is the midpoint of side $BC$ and $D$ is on side $AC$. If the length of $AC$ is 1 and $\angle BAC = 60°, \angle ABC = 100°, \angle ACB = 20°$ and $\angle DEC = 80°$, then the area of $\triangle ABC$ plus twice the area of $\triangle CDE$ equals



18. Let $ABCD$ be a trapezoid with $AB \parallel CD$. The bisectors of $\angle CDA$ and $\angle DAB$ meet at $E$, the bisectors of $\angle ABC$ and $\angle BCD$ meet at $F$, the bisectors of $\angle BCD$ and $\angle CDA$ meet at $G$, and the bisectors of $\angle DAB$ and $\angle ABC$ meet at $H$. Quadrilaterals $EABF$ and $EDCF$ have areas 24 and 36, respectively, and triangle $ABH$ has area 25. Find the area of triangle $CDG$.

19. Consider the set of all triangles $OPQ$ where $O$ is the origin and $P$ and $Q$ are distinct points in the plane with nonnegative integer coordinates $(x, y)$ such that $41x + y = 2009$. Find the number of such distinct triangles whose area is a positive integer.

20. In convex quadrilateral $ABCD, \angle A \cong \angle C, AB = CD = 180$, and $AD \neq BC$. The perimeter of $ABCD$ is 640. Find $\lfloor 1000 \cos A \rfloor$. (The notation $\lfloor x \rfloor$ means the greatest integer that is less than or equal to $x$.)

21. Let $ABCD$ be an isosceles trapezoid, whose dimensions are $AB = 6, BC = 5 = DA$,and $CD = 4$. Draw circles of radius 3 centered at $A$ and $B$, and circles of radius 2 centered at $C$ and $D$. A circle contained within the trapezoid is tangent to all four of these circles. Its radius is $\frac{-k+m\sqrt{n}}{p}$, where $k, m, n$, and $p$ are positive integers, $n$ is not divisible by the square of any prime, and $k$ and $p$ are relatively prime. Find $k + m + n + p$.